Super-Hamiltonian

simple but powerful tool how to solve dynamics in general relativity (for example motion around black hole)

You can even obtain *Mathematica*® code in this talk!*

*This limited-number offer is only available while stock lasts.

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25. November 2015 at 18:00 in SMUF

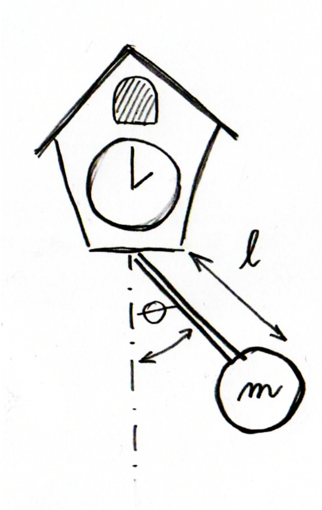
Table of contents / Abstract

After short introduction of Hamiltonian dynamics in classical mechanic, we will try to explore the equation of motion for relativistic particle using covariant formulation of Hamiltonian dynamic. Hamiltonian formalism is:

- simple and mathematically elegant
- very powerful (strong theorems, numerical algorithms)
- provides 'safe ground' in hard-to-understand systems

At the end we will explain few lines of code for particle dynamic written in *Mathematica*® - this code you can take home and use it.

Pendulum - simple dynamics



- one degree of freedom - pendulum angle θ

$$\begin{aligned} F &= ma \\ -mlg\sin(\theta) &= ml^2\theta'' \end{aligned}$$

l pendulum length, m pendulum mass, g grav. constant - they will be normalized '=1'

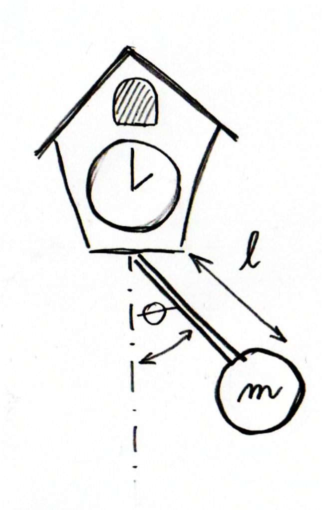
- second order non-linear differential equation

$$\boxed{\theta'' + \sin(\theta) = 0} \tag{1}$$

$$\theta' = \frac{\mathrm{d}\theta}{\mathrm{d}t}, \quad \theta'' = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

- find solution to Eq. (3): $\theta = \theta(t)$???

Pendulum - simple dynamics



- second order non-linear differential equation

$$\theta'' + \sin(\theta) = 0$$

- equivalent to set of first order ODEs

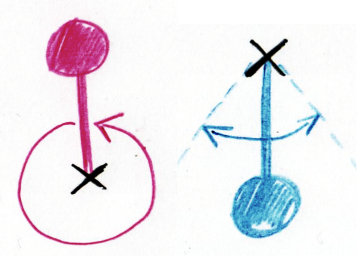
$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}t} &= p, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -\sin(\theta) \end{aligned}$$

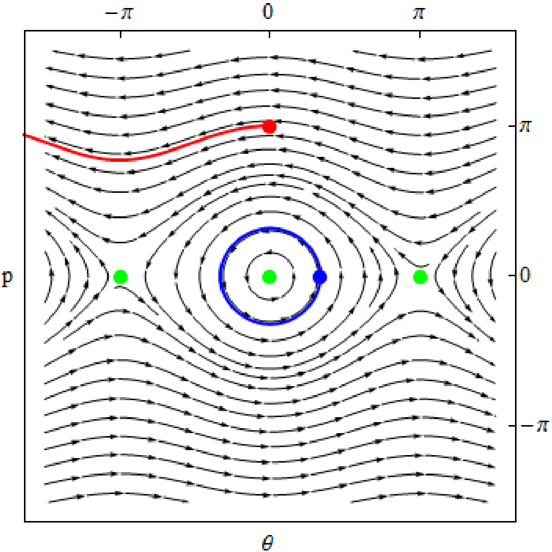
- solution $\theta = \theta(t), p = p(t)$
- two initial conditions (θ_0, p_0) are necessary to find the trajectory
- 2D phase space (every possible state of the system corresponding to one unique point in the phase space)

Pendulum - phase portrait

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = p,$$
$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\sin(\theta)$$

phase space, fixed points, phase curves do not cross each other, prominent (simple) solutions:





"Hamiltonian mechanics is geometry in phase space."

V. I. Arnold, Mathematical methods of classical mechanics, Springer 1989

Hamilton's Principle:

"Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energy."

$$\delta S = 0 \quad \text{for} \quad S = \int_{t_1}^{t_2} (T - U)\, \mathrm{d}t; \quad (T-\text{kinetic}, U-\text{potential})$$

Lagrangian and equations of motion

$$L = L(q_i, \dot{q}_i) = T - U; \qquad \delta S = 0 \rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q} = 0 \qquad \left(L = \frac{1}{2}\dot{\theta}^2 + \cos(\theta) \right)$$

Hamiltonian and equations of motion

$$H = H(q_i, p_i) = T + U; \quad \delta S = 0 \rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i}, \, \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \left(H = \frac{1}{2}p^2 - \cos(\theta) \right)$$

- Why the Hamiltonian mechanic is better then Lagrangian mechanic?

Some features of Hamiltonian mechanics

You are trying to to describe the reality by some physical model:

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\,\mathrm{d}t, \qquad H(q_i, p_i) = \Sigma p_i \dot{q}_i - L(q_i, \dot{q}_i), \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}$$

- Symmetries $\rightarrow$ conserved quantities $\quad H \neq H(q_i) \quad \rightarrow \quad \frac{\partial H}{\partial q_i} = 0$
  p_i does not change (is conserved); q_i is cyclic coordinate
- Dynamical system (chaos)
  - KAM theorem: Integrable Hamiltonian H_0 plus perturbation H_p
    $H = H_0 + \epsilon H_\mathrm{p}$ - most of the tori will survive small perturbation.
- Symplectic integrator - numerical integration scheme where every iteration step is a canonical transformation preserving some invariant (energy) $\rightarrow$ small error in long-term evolution

Dynamics in general relativity

motion in not-curved space + gravity $\sim$ free motion in curved spacetime

- **Curved spacetime** (time 1D + space 3D = spacetime 4D):
  spacetime coordinates x^μ ($\mu = 1, 2, 3, 4$, $x^\mu = (t, x, y, z)$)
  metrics $g_{\alpha\beta}(x)$ - two point distance - line element $\mathrm{d}s^2 = g_{\alpha\beta}(x)\,\mathrm{d}x^\alpha \mathrm{d}x^\beta$
- **Geodesic** $x^\mu = x^\mu(\lambda)$
  is a generalization of a straight line in curved spacetime (=tangent vector $\mathrm{d}x^\mu/\mathrm{d}\lambda$ is parallel transported along this "line" (curve))

$$\frac{\mathrm{D}A^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}A^\mu}{\mathrm{d}\lambda} + \Gamma^\mu{}_{\alpha\beta} A^\alpha \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}, \qquad \Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\rho}(-g_{\alpha\beta,\rho} + g_{\rho\alpha,\beta} + g_{\beta\rho,\alpha})$$

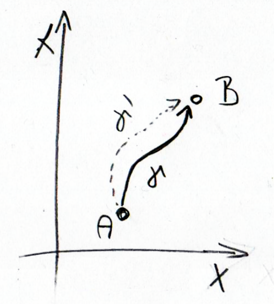
$$\boxed{\frac{\mathrm{D}}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda} + \Gamma^\mu{}_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0}$$

  geodesic is also locally the shortest distance between two points
- **free particles are moving along geodesic**

(norming condition $g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = -1$)

Hamiltonian formalism in general relativity

particle worldline γ (element $\mathrm{d}\gamma$) with parametrization τ



$$S = -m \int_\gamma \mathrm{d}\gamma = -m \int_A^B \sqrt{-h}\,\mathrm{d}\tau$$

induced metric h_{ab}; only 1 worldline coordinate $a, b \in \{\tau\}$

$$h_{ab} = g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}a} \frac{\mathrm{d}x^\beta}{\mathrm{d}b}, \quad h = h_{\tau\tau} = g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}$$

take $\delta S = 0$ and (after some algebra) we have **Super-Hamiltonian**

$$H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + \frac{1}{2} m^2 = 0, \tag{2}$$

and equations of motion $\lambda = \tau/m$

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\partial H}{\partial p_\mu} \quad \Big| = g^{\mu\nu} p_\nu = p^\mu \quad \text{four-momentum}$$

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\lambda} = -\frac{\partial H}{\partial x^\mu} \quad \Big| = g^{\alpha\beta} \Gamma^\gamma{}_{\mu\alpha} p_\gamma p_\beta \quad \text{geodesic}$$

Example of Hamiltonian formalism in GR

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\partial H}{\partial p_\mu}, \qquad \frac{\mathrm{d}p_\mu}{\mathrm{d}\lambda} = -\frac{\partial H}{\partial x^\mu}$$

particle motion around Schwarzschild geometry

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + g_{rr}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2), \qquad -g_{tt} = \frac{1}{g_{rr}} = \left(1 - \frac{2M}{r}\right)$$

- spherical symmetry $\rightarrow$ motion in central plane - $\theta = \pi/2$ $(p_\theta = 0)$
- static geometry - no time t dependence in geometry
  $\partial H/\partial t = 0 \quad \rightarrow \quad -p_t = E,$ energy E is conserved during the motion
- axial symmetry ($\subset$spherical symmetry) - no ϕ dependence in geometry
  $\partial H/\partial \phi = 0 \quad \rightarrow \quad p_\phi = L,$ angular momentum L is conserved

$$H = \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta + \frac{1}{2}m^2 = \frac{1}{2}g^{tt}E^2 + \frac{1}{2}g^{rr}p_r^2 + \frac{1}{2}\frac{1}{r^2}L^2 + \frac{1}{2}m^2 = H(r, p_r)$$

only one degree of freedom $r = r(\lambda)$, phase space 2D $(r \times p_r)$

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\lambda} = m\frac{\mathrm{d}r}{\mathrm{d}\tau} = p^r = g^{rr}p_r, \qquad \frac{\mathrm{d}p_r}{\mathrm{d}\lambda} = -\frac{\partial H}{\partial r}$$

Hamiltonian formalism in GR + electromagnetic field!

$$S = -m \int_\gamma \mathrm{d}\gamma - q \int A_\mu \mathrm{d}x^\mu$$

Gravity $g_{\mu\nu}$: (ex. Schwarzschildblack hole)

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + g_{\phi\phi}\mathrm{d}\phi^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2$$

Electromagnetic field A_α: electromagnetic four-potential $A^\mu = (\phi, A^i)$; ϕ is the electric potential, A^i is the magnetic potential (vector potential).

Super-Hamiltonian and equation of motion

$$H = \frac{1}{2} g^{\alpha\beta}(\pi_\alpha - qA_\alpha)(\pi_\beta - qA_\beta) + \frac{1}{2}\, m^2,$$

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\zeta} \equiv p^\mu = \frac{\partial H}{\partial \pi_\mu}, \quad \frac{\mathrm{d}\pi_\mu}{\mathrm{d}\zeta} = -\frac{\partial H}{\partial x^\mu}$$

kinematical four-momentum $p^\mu = m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$; generalized (canonical) $\pi^\mu = p^\mu + qA^\mu$

Hamiltonian formalism in GR - strings, strings!

as the string moves it sweeps 2D worldsheet w - the worldsheet area should be minimal

$$S = -\mu \int_w \mathrm{d}w = -\mu \int \sqrt{-h}\, \mathrm{d}\sigma \mathrm{d}\tau$$

worldsheet coordinates: τ - string evolution and σ - along string; $a, b \in \{\tau, \sigma\}, \mu \in \{t, x, y, z\}$

induced metric on the worldsheet and worldsheet stress-energy tensor (density)

$$h_{ab} = g_{\alpha\beta} x^{\alpha}_{|a} x^{\beta}_{|b}, \quad \Sigma^{ab} = -\mu\sqrt{-h}h^{ab}$$

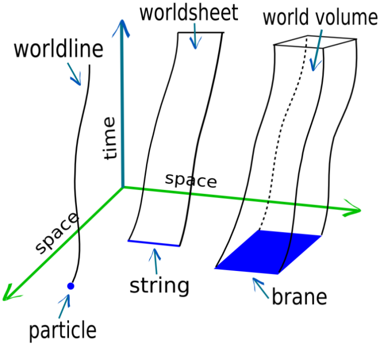


Fig. created by Stevertigo is taken from Wikipedia

take $\delta S = 0$ and (after some algebra) string Hamiltonian and equation of motion

$$H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + f(\Sigma^{ab}); \qquad \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\partial H}{\partial p_\mu}, \quad \frac{\mathrm{d}p_\mu}{\mathrm{d}\lambda} = -\frac{\partial H}{\partial x^\mu}$$

...and scalar field can "live" on string $\quad S = -\int (\mu + h^{ab} \varphi_{|a} \varphi_{|b}) \sqrt{-h}\, \mathrm{d}\sigma \mathrm{d}\tau$

Few lines of code in *Mathematica*®

```
gtt[r_, θ_] := -AA[r]; grr[r_, θ_] := 1/AA[r]; gθθ[r_, θ_] := r^2; gϕϕ[r_, θ_] := r^2 Sin[θ]^2; (* lower indicies *)
gUtt[r_, θ_] := 1/gtt[r, θ]; gUrr[r_, θ_] := 1/grr[r, θ]; gUθθ[r_, θ_] := 1/gθθ[r, θ]; gUϕϕ[r_, θ_] := 1/gϕϕ[r, θ]; (* upper indicies *)
(*---- Metrics coefficients ----*)
EE = Sqrt[EB2[R0, θ0, BB, LL]];
(*---- Hamiltonian a equations of motion ----*)
HoHo[r_, θ_, pr_, pθ_] := 1/2 gUrr[r, θ] pr^2 + 1/2 gUθθ[r, θ] pθ^2 + 1/2 gUtt[r, θ] EE^2 + 1/2 gUϕϕ[r, θ] (LL - BB gϕϕ[r, θ])^2 + 1/2;
HamI = HoHo[r[τ], θ[τ], pr[τ], pθ[τ]]; ERR[r_, θ_, pr_, pθ_] := Abs[HoHo[r, θ, pr, pθ]]; (* invariant for numerical integration and error f

DrHoHo[r_, θ_, pr_, pθ_] := ∂r HoHo[r, θ, pr, pθ]; DθHoHo[r_, θ_, pr_, pθ_] := ∂θ HoHo[r, θ, pr, pθ];

Equations = {
  r'[τ]  ==  gUrr[r[τ], θ[τ]] pr[τ],  pr'[τ] == -DrHoHo[r[τ], θ[τ], pr[τ], pθ[τ]],
  θ'[τ]  ==  gUθθ[r[τ], θ[τ]] pθ[τ],  pθ'[τ] == -DθHoHo[r[τ], θ[τ], pr[τ], pθ[τ]],
  ϕ'[τ]  ==  (gUϕϕ[r[τ], θ[τ]] LL) - BB (*This component is given by  \dot{\phi} / \tau = P^\phi = g^{\phi\phi} P_{\phi} = g^{\phi\phi}
 };
IntCond = {r[0] == R0, pr[0] == 0, θ[0] == θ0, pθ[0] == 0, ϕ[0] == 0};
(*---- Hamiltonian a equations of motion ----*)

(*---- trajectory calculation ----*)
Metoda = {"StiffnessSwitching", Method → {{"Projection", Method → {"ExplicitRungeKutta", "DifferenceOrder" → 8}, "Invariants" → HamI}, Aut
(* put your favourite numerical method here *)

draha = NDSolve[{Equations, IntCond}, {r, pr, θ, pθ, ϕ}, {τ, 0, τEND},
  {Method → {"EventLocator", "Event" → {r[τ] - Rend, r[τ] - xInfinity}, "EventAction" :→ {Throw[Null, "StopIntegration"], Throw[Null, "StopI
   {StartingStepSize → 10^-1, MaxSteps → ∞}}];
ipub = r /. draha[[1, 1]]; domain = ipub["Domain"]; {begin, end} = domain[[1]];
```

http://nora.fpf.slu.cz/~kolos/data/hamiltonian.nb

Summary and some reference

Hamiltonian formalism is

- simple and mathematically elegant
- used in different fields of physics - lot of examples
- dynamical systems (chaos)
- symplectic integrator - small error in long-term evolution
- provides 'safe ground' in hard-to-understand systems

Thank you for your attention

`http://nora.fpf.slu.cz/~kolos/data/hamiltonian.nb`

C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, San Francisco: W.H. Freeman and Co., (1973)
B. Carter and D. A. Steer, *Symplectic structure for elastic and chiral conducting cosmic string models*, Physical Review D, vol. **69**, Issue 12, (2004), arXiv:hep-th/0307161