Transition from regular to chaotic motion

in the neighborhood of stable equilibrium point (relativistic string loop case)

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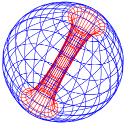


Outline

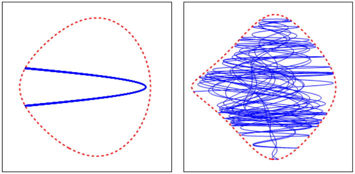
Dynamical systems

- Problem specification
- Harmonic oscillator
- KAM and other theorems
- How to measure chaos?

Application to string motion

- String loop model
- Normal form construction
- Non-linearity for individual trajectory
- String "focusing" problem
- Grow of the chaos with energy increase

Problem specification



- Integrable Hamiltonian H_0 plus perturbation H_p

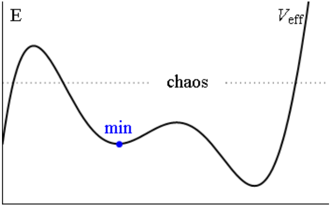
$$H = H_0 + \epsilon H_\mathrm{p}$$

As the ϵ increase, it cause a non-linearity in the system.

- Not every system can be splitted into "regular" + "perturbation"

$$H = H_\mathrm{d}(\mathbf{p}, \mathbf{q}) + V_{\mathit{eff}}(\mathbf{q}),$$

there can exist minima in "effective potential" $V_{\mathit{eff}}(\mathbf{q})$ with energy E_0. Now increase of energy $\triangle E$ above E_0 cause increase of non-linearity.



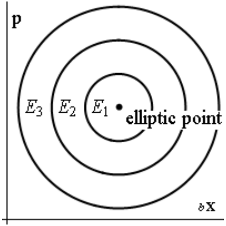
Example of integrable systems

1DOF & 2DOF harmonic oscillator, fundamental frequencies

- 1DOF harmonic oscillator, phase space 2D

  $$H = 1/2\left(p^2 + \omega^2 x^2\right) \quad \Rightarrow \quad \ddot{x} + \omega^2 x = 0$$

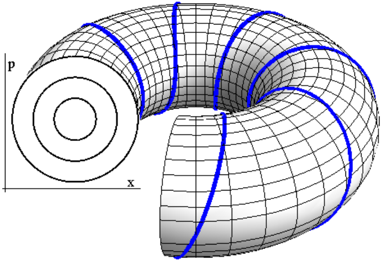
  trajectory lies on S_1 for every energy level



- 2DOF harmonic oscillator

  $$\begin{aligned} H &= 1/2\left(p_x^2 + \omega_x^2 x_1^2\right) \\ &+ 1/2\left(p_y^2 + \omega_y^2 y_2^2\right) \end{aligned}$$

  4D phase space (x, p_x, y, p_y)
  trajectory lies on torus $S_1 \times S_1$
  fundamental frequencies ω_x, ω_y



Theorems from theory of dynamical systems

$$H = H_0 + \epsilon H_{\mathrm{p}}$$

- Kolmogorov-Arnold-Moser (KAM) theory - Most of the tori will survive small perturbation.
- Birkhoff normal form - The Birkhoffs theorem ensures the existence of a canonical transformation putting a Hamiltonian system in normal form up to a remainder of a given order.
- KAM around elliptic point (minima in eff. potential) - for 2 DOF:

$$k_1\,\omega_1 + k_2\,\omega_2 = 0, \quad k_1 + k_2 < 4$$

for resonances **1:1, 1:2, 2:1** we can't construct normal forms.

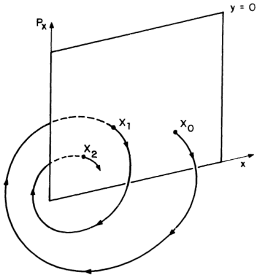
System will oscillate in a quasi-periodic motion, if the parameter ϵ remains small. As the parameter ϵ grows, the condition $\epsilon << 1$ becomes violated, the nonlinear parts in the Hamiltonian become stronger, and we enter the nonlinear, chaotic regime of its motion.
Increase of non-linearity of a system moving in vicinity of its local stable equilibrium point (minimum) is caused by increase of its energy.

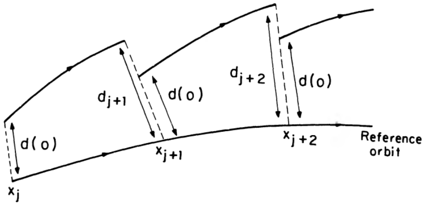
How to measure chaos?

Power Spectrum, Poincare sections, Maximal Lyapunov exponent

- Power Spectrum - Fourier transformation
- Poincare sections
- Maximal Lyapunov exponent

$$\lambda_{\mathrm{L}} = \lim_{\substack{d_0 \to 0 \\ t \to \infty}} \left(\frac{1}{t} \ln \left(\frac{d(t)}{d_0} \right) \right)$$



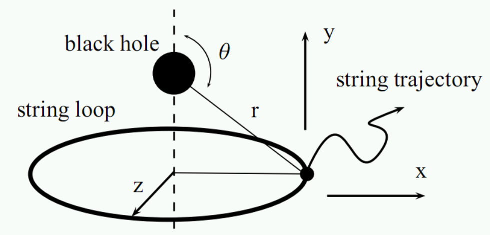


Lyapunov exponent is describing the two orbits separation and hence the measure of chaos.

(Problem: Maximal Lyapunov exponent in general relativity)

String loop introduction

Current-carrying string loop and BH



- string loop threaded on to axis of black hole
- string oscillate in x-z plane, propagating in y direction.
- string tension has μ (prevents expanding) & scalar field φ crates overall current J - creates angular momentum (prevents collapsing)

$$H = \frac{1}{2} g^{rr} P_r^2 + \frac{1}{2} g^{\theta\theta} P_\theta^2 + \frac{1}{2} g^{tt} E^2 + \frac{1}{2} g^{rr} \left(r \sin\theta + \frac{J^2}{r \sin\theta} \right)^2$$

- Hamiltonian - dynamical part (red), effective potential (blue).

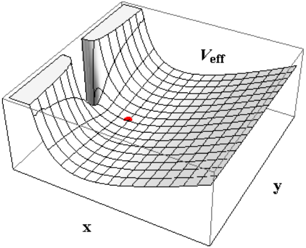
Normal form construction

Construction of normal form in the minima of effective potential

- Effective potential V_{eff} (Schwarzschild):

$$V_{\text{eff}}(r,\theta) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(x + \frac{J^2}{x}\right)^2$$

Minima is located at $X_0^\alpha = (r_0, \theta_0)$



- New coordinates and momenta $X^\alpha = X_0^\alpha + \epsilon \hat{X}^\alpha, \quad P_\alpha = \epsilon \hat{P}_\alpha$

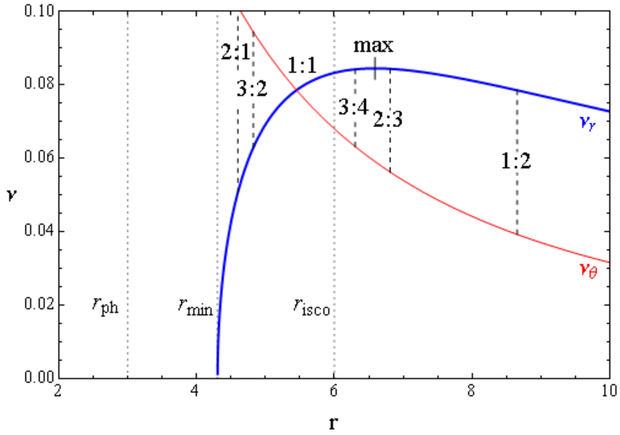
$$H(\hat{P}_\alpha, \hat{X}^\alpha) = H_0 + \epsilon H_1(\hat{X}^\alpha) + \epsilon^2 H_2(\hat{P}_\alpha, \hat{X}^\alpha) + \epsilon^3 H_3(\hat{P}_\alpha, \hat{X}^\alpha) + \dots$$

rescale energy at min.: $H_0 = 0$ & local minimum: $H_1(\hat{X}^\alpha) = 0$

$$H = {}^1\!/_2 \left(\hat{p}_r^2 + \omega_r^2 \hat{r}^2\right) + {}^1\!/_2 (\hat{p}_\theta^2 + \omega_\theta^2 \hat{\theta}^2) + \epsilon H_3(\hat{P}_\alpha, \hat{X}^\alpha) + \dots$$

is Hamiltonian in the vicinity of the local minimum

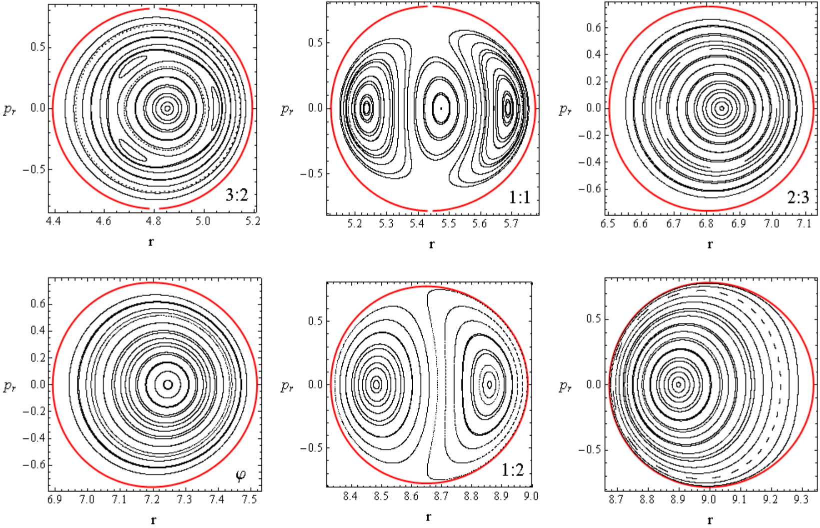
Fundamental frequencies



$$H = {}^1\!/_2 \left(\hat{p}_r^2 + \omega_r^2 \hat{r}^2\right) + {}^1\!/_2 (\hat{p}_\theta^2 + \omega_\theta^2 \hat{\theta}^2); \quad \omega_r^2 = \frac{r^2 - 5r + 3}{r^4}, \;\; \omega_\theta^2 = \frac{1}{r^3}$$

"problematic" resonance radii: 1:1 ($r \sim 5.5$), 1:2 ($r \sim 8.7$); (2:1 too shallow)

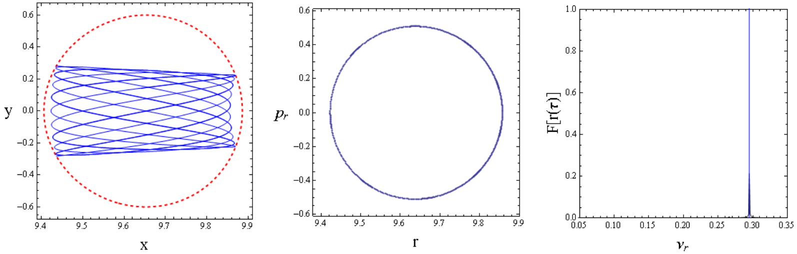
Fate of resonant & non-resonant torii

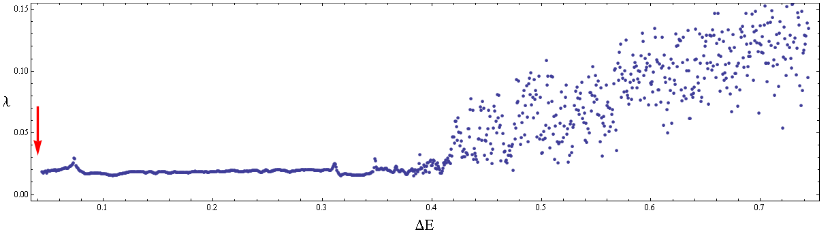


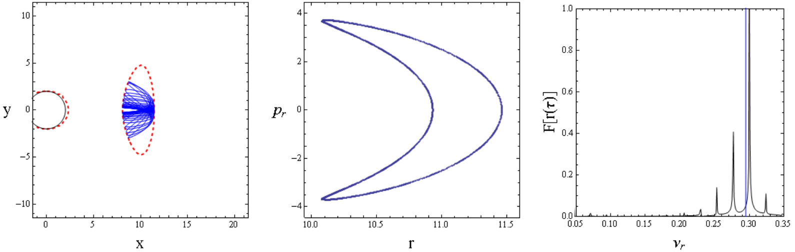
Energy $\triangle E = 0.01$ & only tori for resonance 1:1, 1:2, (2:1) are destroyed

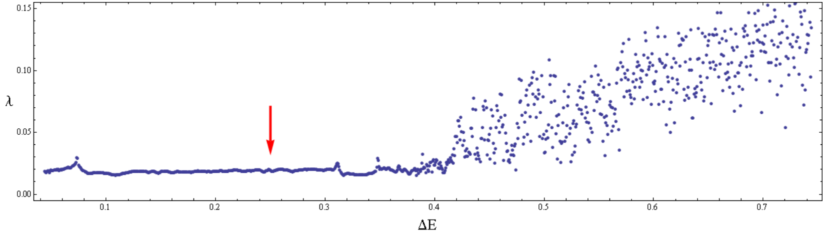
Growth of non-linearity for individual trajectory

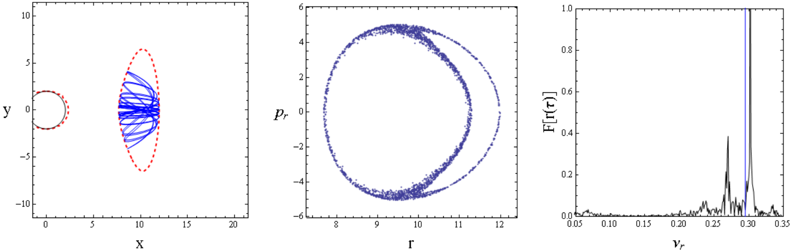
Trajectory / Poincare section / Power Spectrum

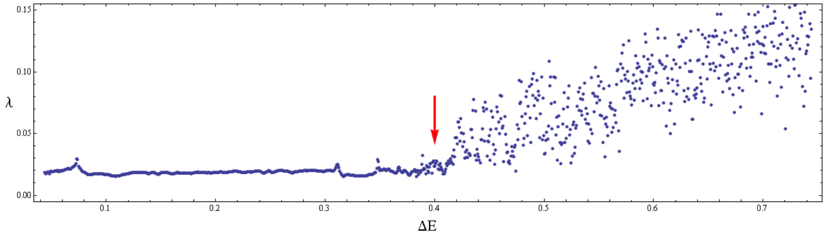


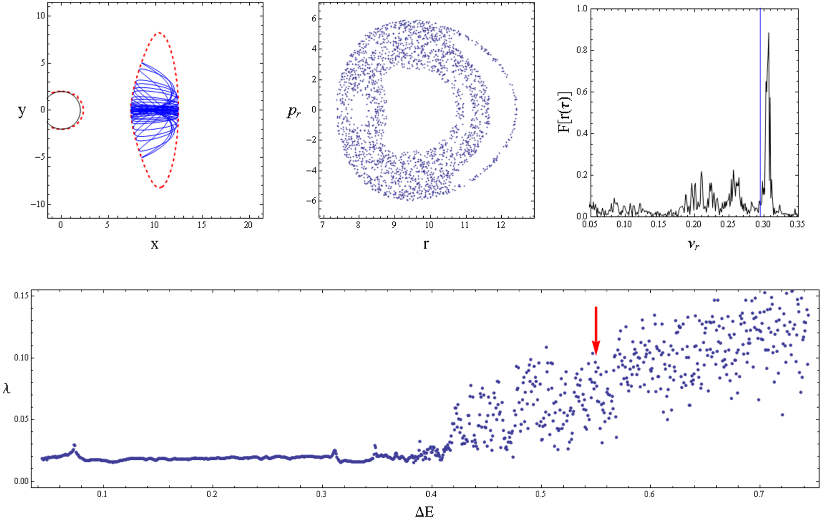



y
x
10
5
0
−5
−10
0
5
10
15
20
p_r
r
4
2
0
−2
−4
10.0
10.5
11.0
11.5
F[r(τ)]
ν_r
1.0
0.8
0.6
0.4
0.2
0.0
0.05
0.10
0.15
0.20
0.25
0.30
0.35

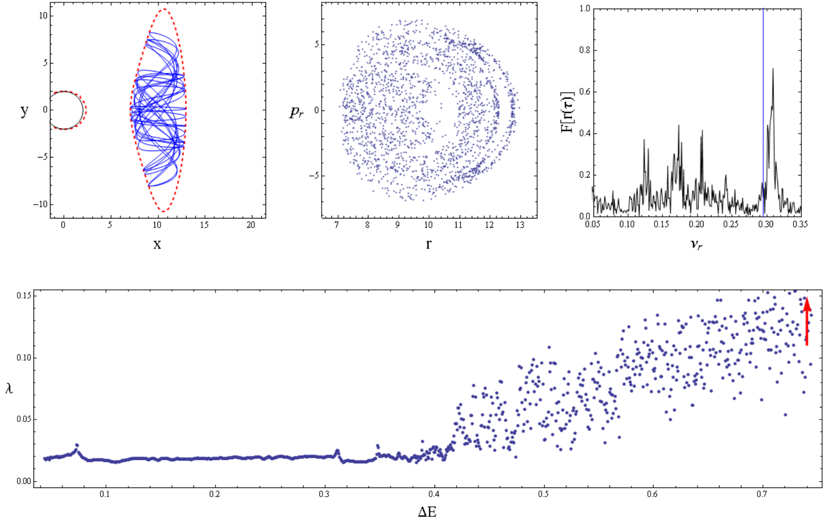

λ
ΔE
0.15
0.10
0.05
0.00
0.1
0.2
0.3
0.4
0.5
0.6
0.7


y
x
p_r
r
F[r(τ)]
ν_r

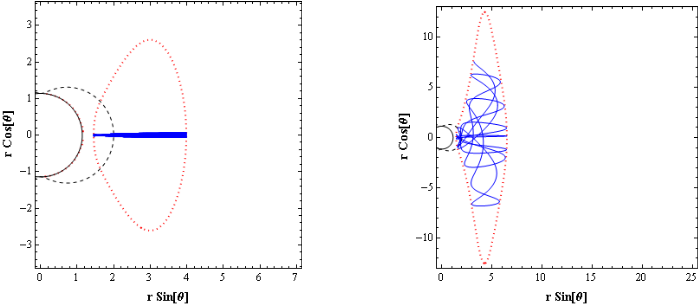

λ
ΔE



y
x
p_r
r
F[r(τ)]
ν_r
λ
ΔE

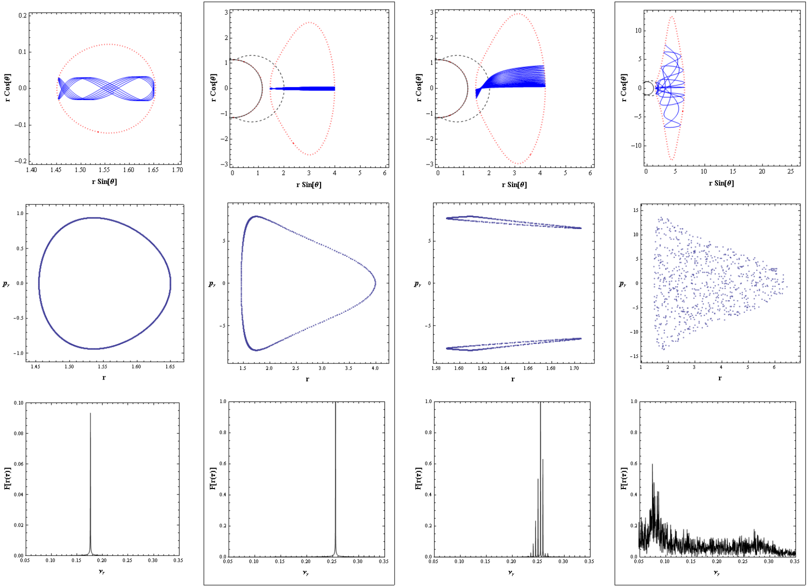

y
x
p_r
r
F[r(τ)]
ν_r
λ
ΔE

String "focusing" problem

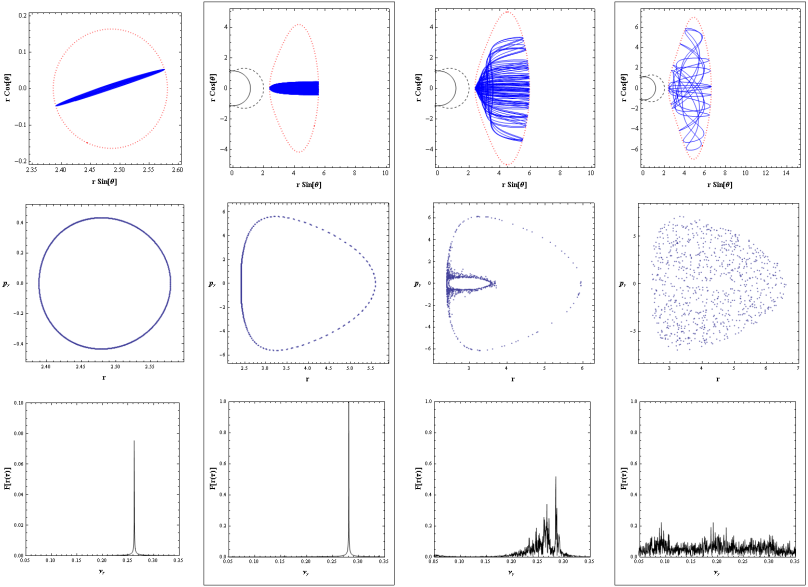


(a) Regular ($E = 5.5$)   (b) Chaotic ($E = 8$)

- Transition from the regular to the chaotic regime of the motion is the solution to the "focusing" problem of the string trajectories from T. Jacobson and T. P. Sotiriou:
  String dynamics and ejection along the axis of a spinning black hole

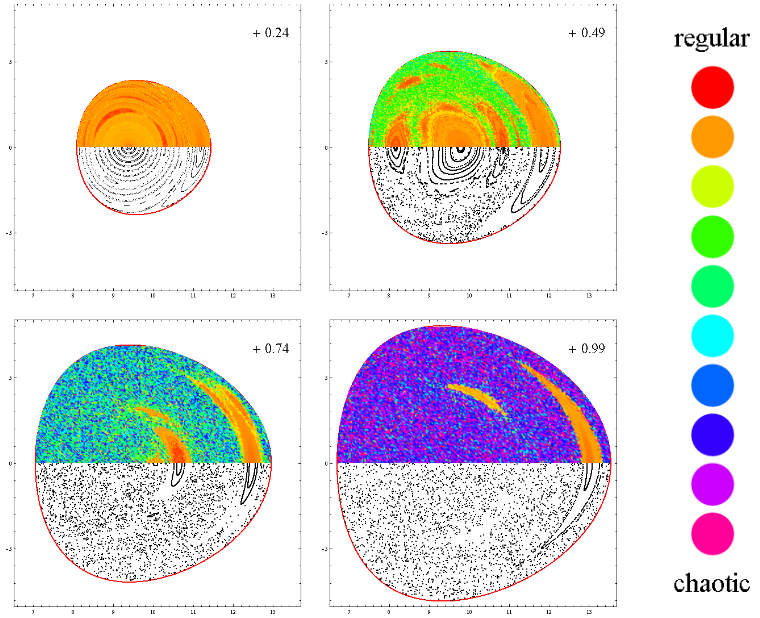


(c) $E = 3.3$ (d) $E = 5.5$ (e) $E = 5.7$ (f) $E = 8$



(g) $E = 5$ (h) $E = 8$ (i) $E = 8.35$ (j) $E = 9$

Chaos is comming. . .



Summary and future work

- Increase of non-linearity is caused by increase of energy
- Explanation of string "focusing" problem

- Estimation of transition energy (regular/chaotic)
- Behavior at resonance radii

Thank you for your attention.

This presentation will be found at http://www.physics.cz/research/